Quantum speed-ups for sieving algorithms for the shortest vector problem

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based on joint work with Erik Mårtensson, Eamonn W. Postlethwaite, Subhayan Roy Moulik

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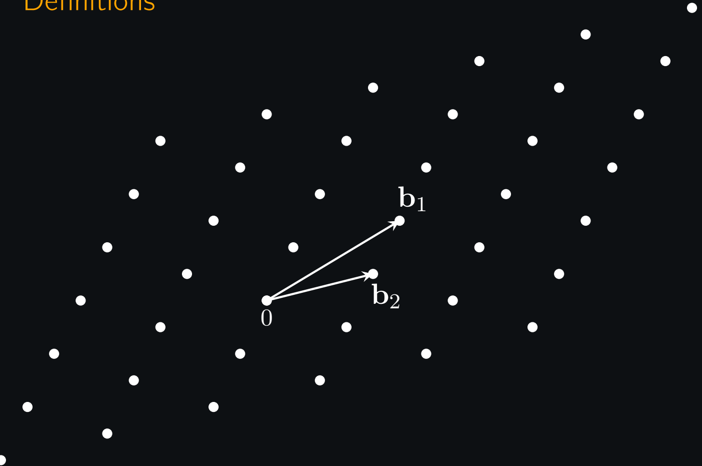
June 11, 2020

Definitions



A lattice is a set $\mathcal{L} = \{\sum_{i \le n} x_i \mathbf{b}_i \ : \ x_i \in \mathbb{Z}\}$ for some linearly independent $\mathbf{b}_i \in \mathbb{R}^n$

$\{\mathbf{b}_i\}_i$ – a basis of $\mathcal{L}$

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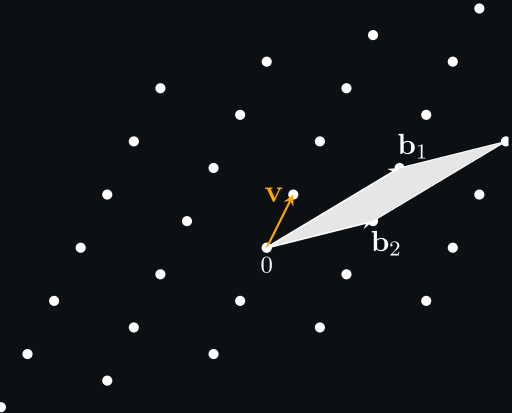

Minimum

$$\lambda_1(\mathcal{L}) = \min_{\mathbf{v}\in\mathcal{L}\setminus\mathbf{0}} \|\mathbf{v}\|$$

Determinant

$$\det(\mathcal{L}) = |\det(\mathbf{b}_i)_i|$$

Minkowski bound

$$\lambda_1(\mathcal{L}) \leq \sqrt{n}\cdot\det(\mathcal{L})^{\frac{1}{n}}$$

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SVP

The Shortest Vector Problem (SVP) asks to find $\mathbf{v}_{\text{shortest}} \in \mathcal{L}$:

$$\|\mathbf{v}_{\text{shortest}}\| = \lambda_1(\mathcal{L})$$

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Often we are satisfied with an approximation (γ-SVP) to $\mathbf{v}_{\text{shortest}}$:

$$||\mathbf{v}_{\text{short}}|| \leq \gamma \cdot \lambda_1(\mathcal{L})$$

Why is SVP interesting?

$$||\mathbf{v}_{\text{short}}|| \leq \gamma \cdot \lambda_1(\mathcal{L})$$

Hardness of (approx)-SVP underlies all lattice-based cryptographic constructions.

- For $\gamma = 2^{\log^{1-\varepsilon} n}$ SVP is NP-hard
- Crypto is based on $\gamma = \mathrm{poly}(n)$
- We assume SVP is infeasible for $n > 350$
- What we can achieve now is $n = 170$ using lots of RAM and GPUs, see TU Darmstadt's SVP challenge[1]
- There is an open-source library G6K[2] for solving SVP

[1] https://www.latticechallenge.org/svp-challenge/
[2] https://github.com/fplll/g6k

Asymptotics ($+o()$ everywhere) of γ-SVP, $\gamma < \mathrm{poly}(n)$, $n := \dim \mathcal{L}$

Classical	Quantum

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Enumeration

$\log \mathsf{Time} = \frac{1}{2e} n \log n$	$\log \mathsf{Time} = \frac{1}{4e} n \log n$
$\mathsf{Mem} = \mathrm{poly}(n)$ [HS07]	$\mathsf{Mem} = \mathrm{poly}(n)$ [ANS18]

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Based on discrete Gaussian samplers	
$\log \text{Time} = 1n$	$\log \text{Time} = 1.2553n$
$\log \text{Mem} = 1n$ [ADRS15]	$\log \text{Mem} = 0.5n$ [LLK18]

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Sieving (heuristic) +LSH	
$\log \text{Time} = 0.292n$	$\log \text{Time} = 0.265n$
$\log \text{Mem} = 0.208n$ [BDGL16]	$\log \text{Mem} = 0.265n$ [Laa16]
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$\log \text{Time} = 0.396n$	$\log \text{Time} = 0.299n$
$\log \text{Mem} = 0.189n$ [HK17]	$\log \text{Mem} = 0.139n$ [KMPR19]

Time/Memory trade-offs exist

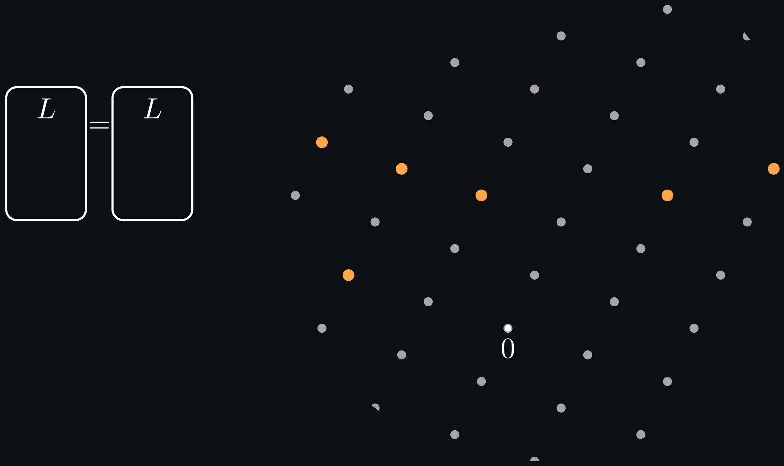
For quantum algorithms Mem means quantumly addressable classical RAM.

Sieving for SVP

https://sites.google.com/a/x.bestledlights.cf/a223/1-32772214555

Basic 2-Sieve (Nguyen-Vidick sieve)

Main idea in all sieving algorithms: saturate space with enough lattice vectors so that their sums give short(er) vectors



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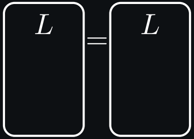

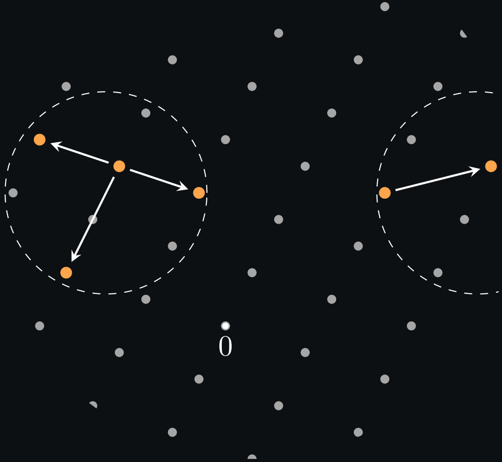

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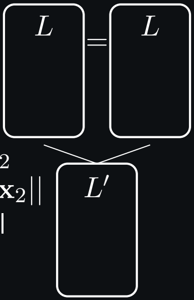

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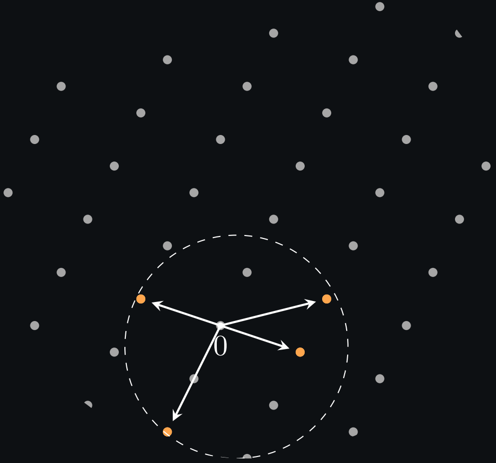
is small



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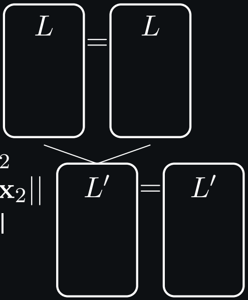

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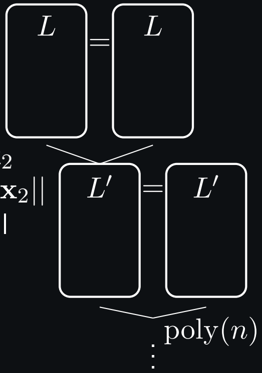

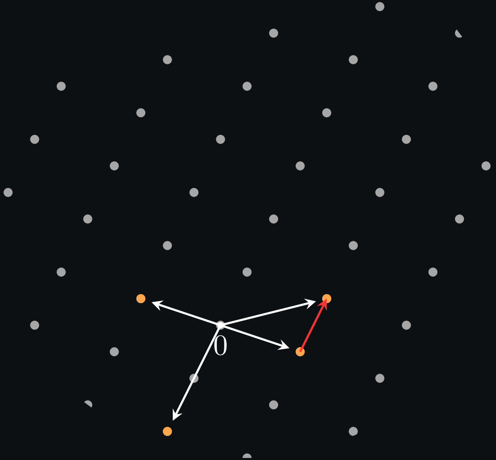

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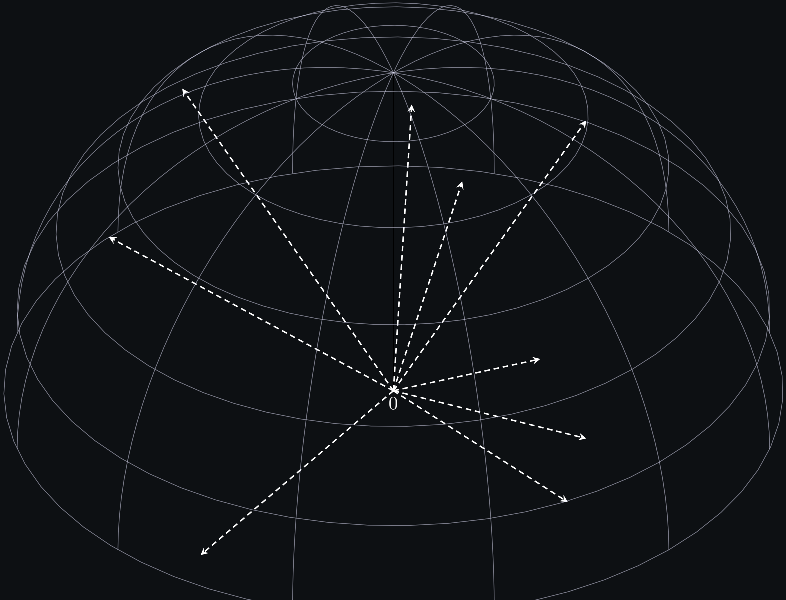
Assumption: vectors (normalized) in $|L|$ are uniform iid on S^{n-1}.



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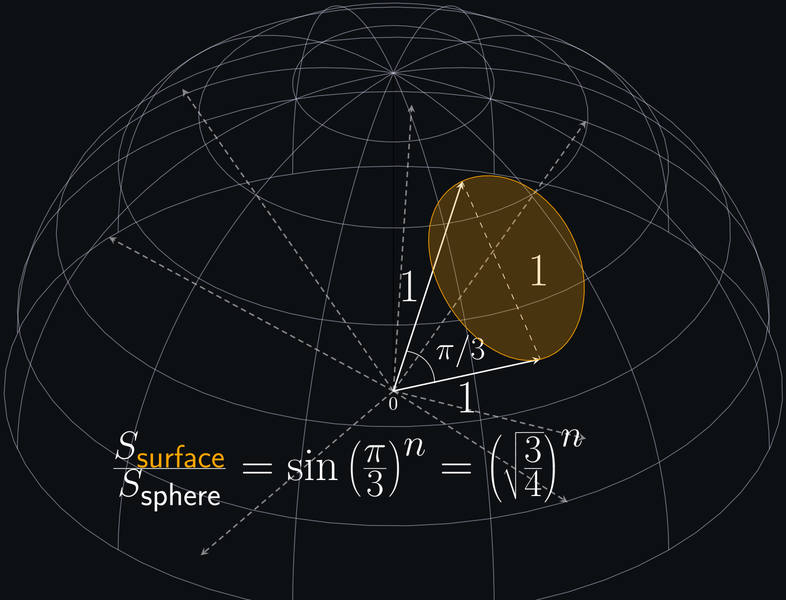

$$\frac{S_{\text{surface}}}{S_{\text{sphere}}} = \sin\left(\frac{\pi}{3}\right)^n = \left(\sqrt{\frac{3}{4}}\right)^n$$

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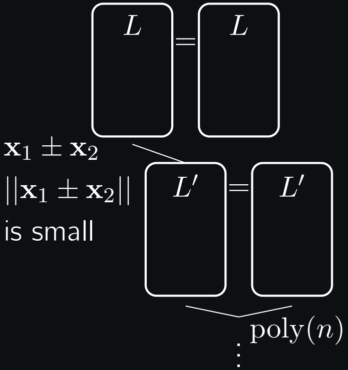
Main idea in all sieving algorithms: saturate space with enough lattice vectors so that their sums give short(er) vectors



$$\text{Mem} = \left(\sqrt{\frac{3}{4}}\right)^{-n} = 2^{0.2075n}$$

$$\text{Time}^{\text{Class}} = |L|^2 = 2^{0.415n}$$

All $o(n)$ terms are omitted

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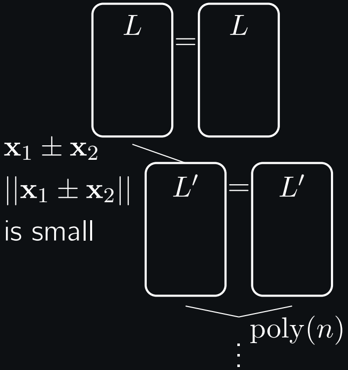

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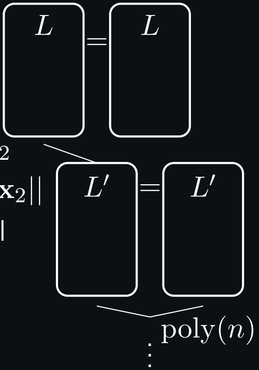

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- we need to find almost all $((1-o(1))$–fraction of) close pairs
- 'close' pairs are much closer than the two random ones

All $o(n)$ terms are omitted

How to improve classical runtime to $T = 2^{0.292n+o(n)}$?

Use Near Neighbour search!

Locality-sensitive filtering [BGJ15, BDGL16]

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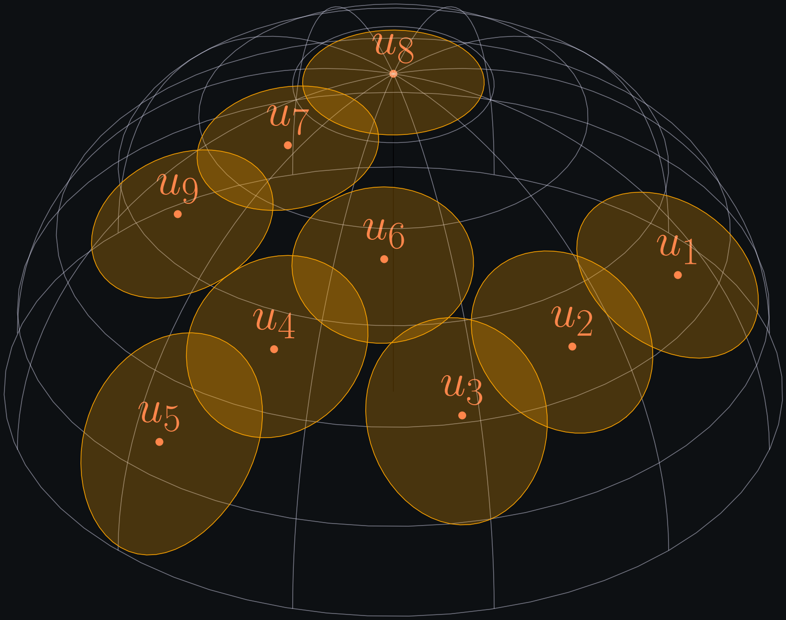
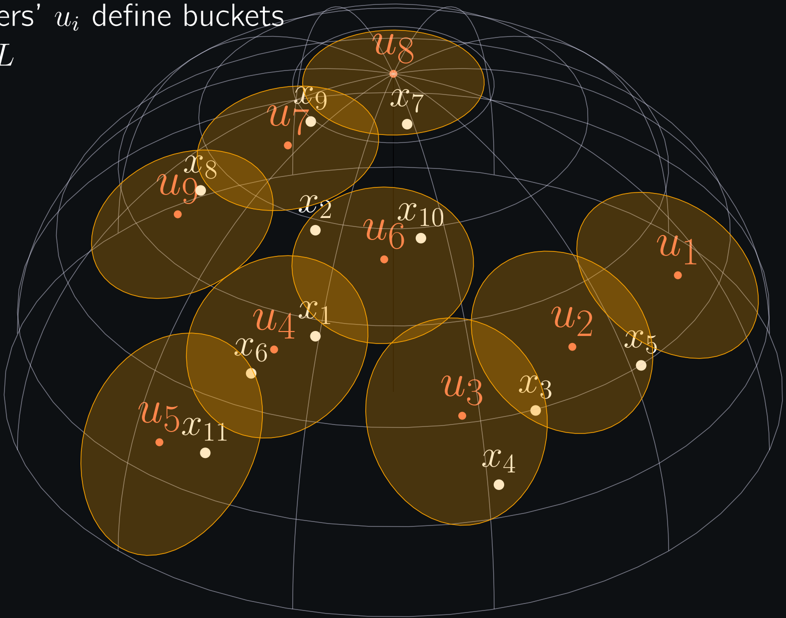


Locality-sensitive filtering [BGJ15, BDGL16]

'centers' u_i define buckets

$x_i \in L$



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For all $\mathbf{u}_i$:
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        If $|\langle \mathbf{x}, \mathbf{u}_i \rangle|$ is large enough
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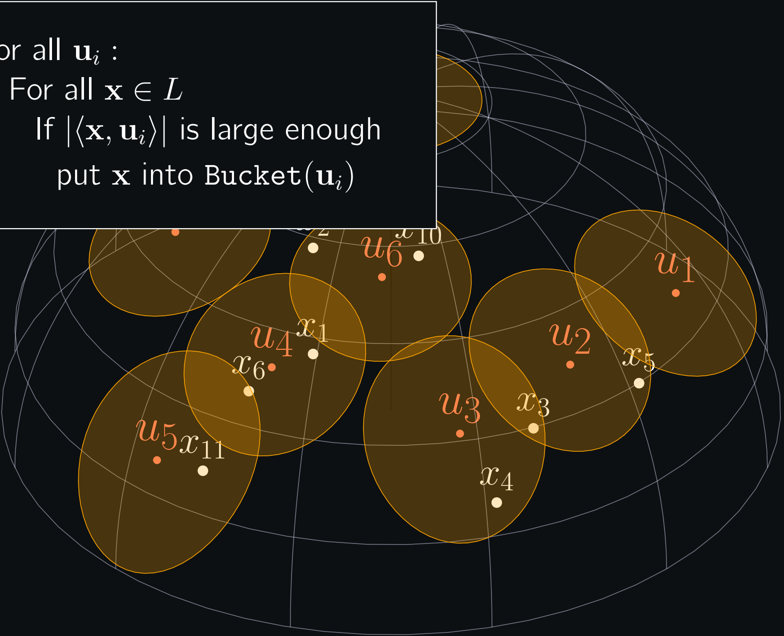

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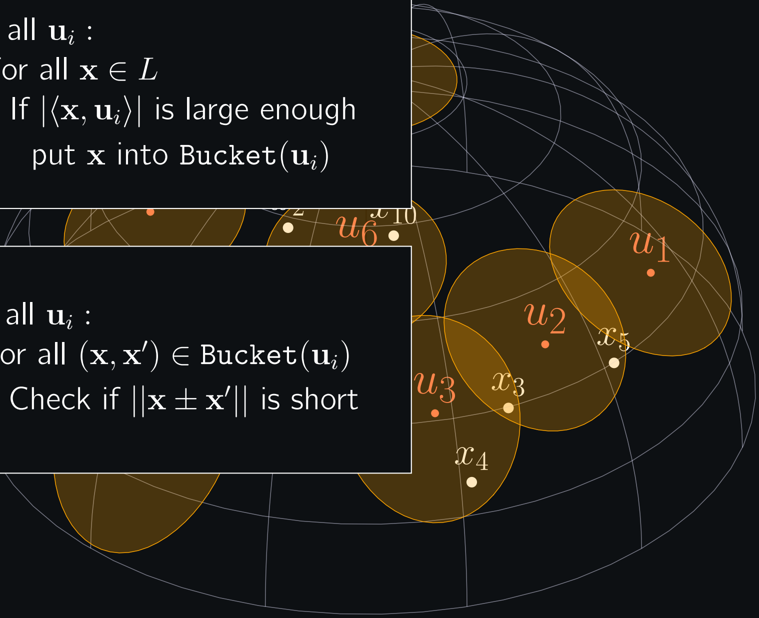

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For $2^{(0.142+o(1))n}$ many $\mathbf{u}_i$'s:

$$T = 2^{(0.349+o(1))n}$$

When $\mathbf{u}_i$'s are of special form

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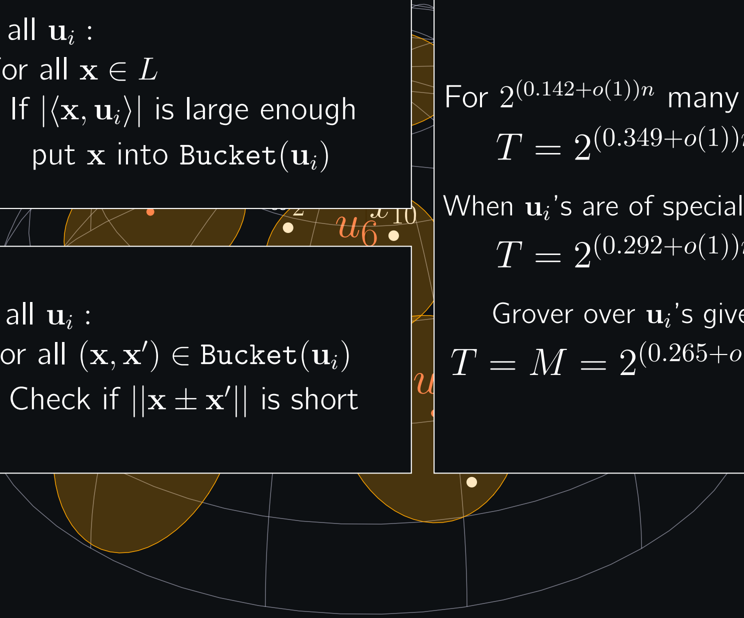

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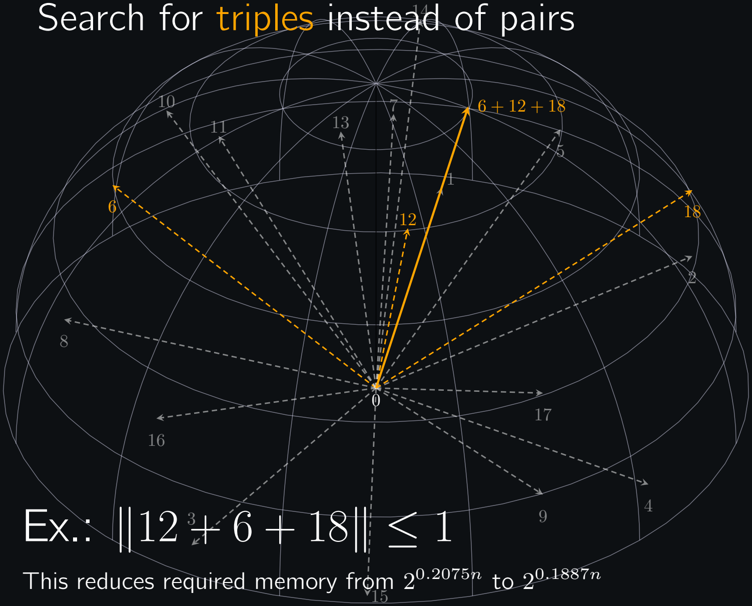
Lower memory?

3-Sieve as 3-List problem

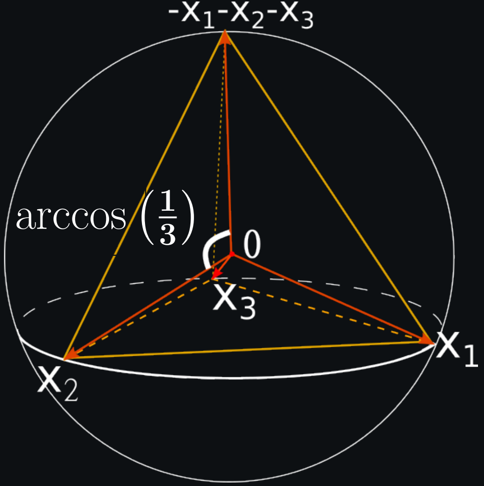
Search for triples instead of pairs



Ex.: $\|12 + 6 + 18\| \leq 1$

This reduces required memory from $2^{0.2075n}$ to $2^{0.1887n}$

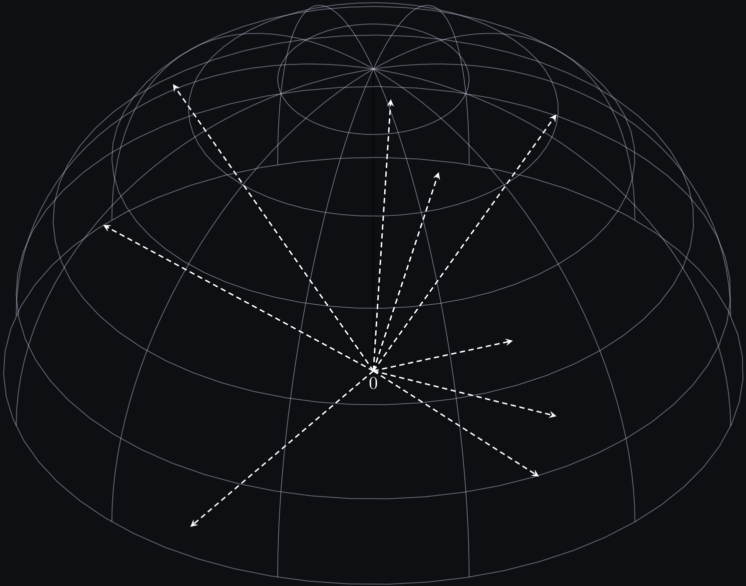
Configuration of good triples, [HK17]



All good triples are concentrated in the shape of 3-simplex

3-Sieve via triangle finding

Connect two points $(i, j) \Leftrightarrow |\langle i \, , j \rangle| \approx 1/3$



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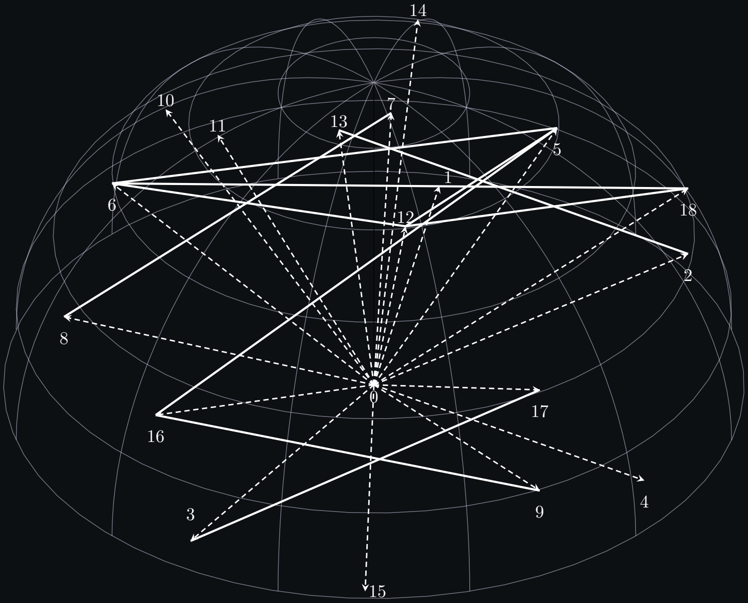

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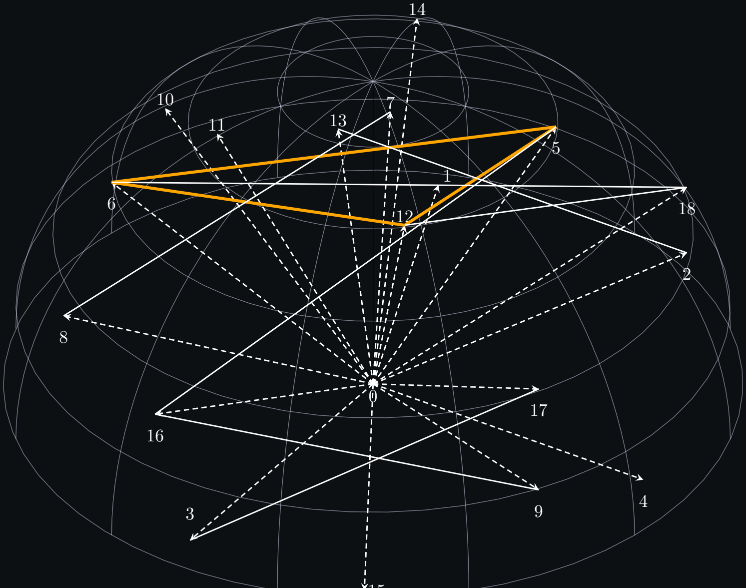
Good triples $(i, j, k) \Leftrightarrow$ triangles



Apply quantum triangle (k-clique) finding

$G = \{V, E\}$, V – lattice vectors, $e(v_i, v_j) \in E \Leftrightarrow |\langle v_i\ , v_j\rangle| \approx 1/3$

Run triangle listing on G (it's a sparse graph!)

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Adapt the triangle **finding** algorithm of [Buhrman–de Wolf-Dürr–Heiligman–Høyer–Magniez–Santha]:

Time (find $\triangle$) $= \sqrt{|E|} \implies$ Time (list all $\triangle's$) $= |V|\sqrt{|E|}$

This gives

$$\text{Time}^{\texttt{Quant}} = 2^{0.335n} \quad cf. \quad \text{Time}^{\texttt{Class}} = 2^{0.396n}$$

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The algorithm generalises to larger $k = \Theta(1)$ and time-optimal inner product leading to

$$\text{Time}^{\texttt{Quant}} = 2^{0.299n + o(n)} \qquad \text{Memory}^{\star} = 2^{0.139n + o(n)}$$

$\star$ quantumly addressable classical memory

More results and conclusions

- Overall now we have
  - ✓ best $\texttt{Time} \times \texttt{Area}$

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cf.

$$\text{Time}^{\texttt{Class}} = 2^{0.373n+o(n)} \qquad \text{Memory} = 2^{0.186n+o(n)}$$

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Thank you!

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References I

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