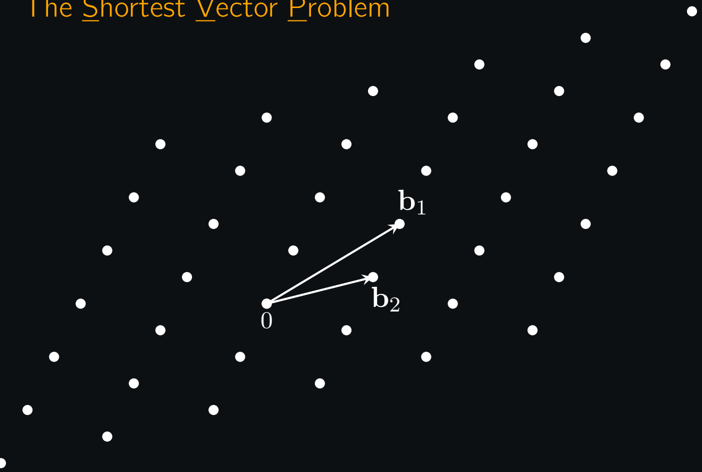
Quantum time-memory trade-offs for lattice sieving algorithms

Elena Kirshanova

based on joint work with Erik Mårtensson, Eamonn W. Postlethwaite, Subhayan Roy Moulik

Dagstuhl, Germany
October 15, 2019

The Shortest Vector Problem



A lattice is a set $\mathcal{L} = \{\sum_{i \le n} x_i \mathbf{b}_i \ : \ x_i \in \mathbb{Z}\}$ for some linearly independent $\mathbf{b}_i \in \mathbb{R}^n$

$\{\mathbf{b}_i\}_i$– a basis of $\mathcal{L}$

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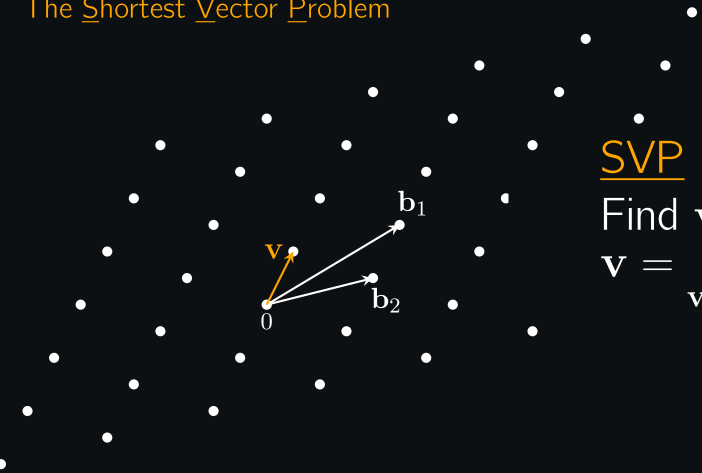

SVP

Find $\mathbf{v} \in \mathcal{L}$:

$$\mathbf{v} = \min_{\mathbf{v} \neq 0 \in \mathcal{L}} \|\mathbf{v}\|$$

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Asymptotical ($+o()$ everywhere) Hardness of SVP. $n := \dim \mathcal{L}$

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$\log \text{Time} = \frac{1}{2e} n \log n$		$\log \text{Time} = \frac{1}{4e} n \log n$
$\text{Mem} = \text{poly}(n)$		

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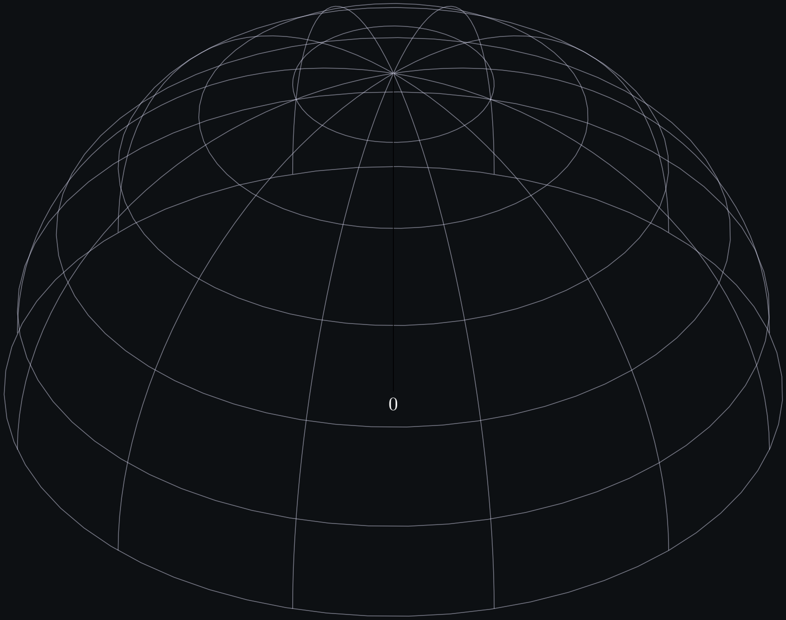

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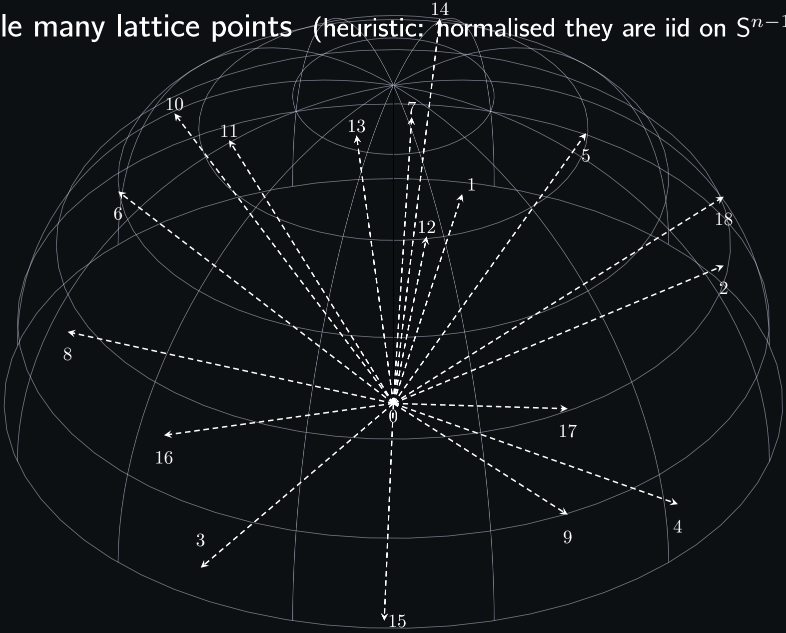

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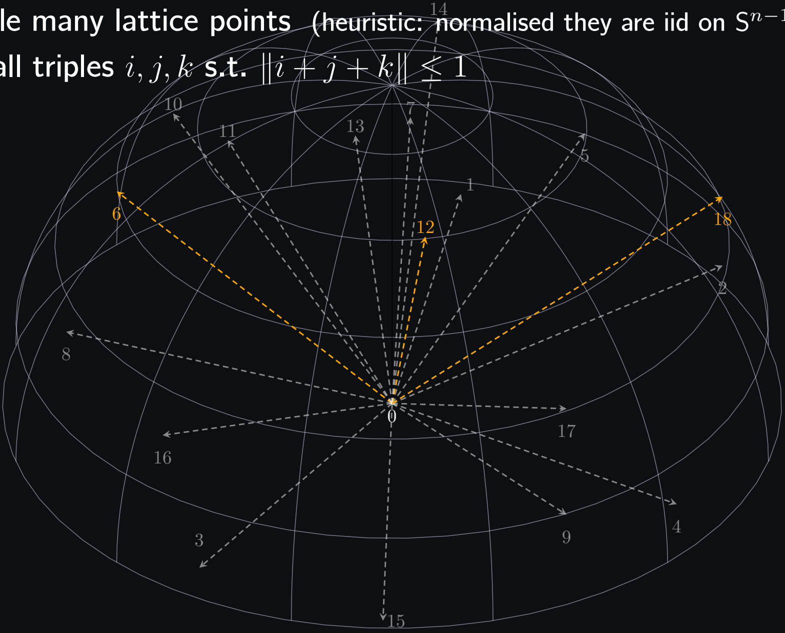
Find all triples i, j, k **s.t.** $\|i + j + k\| \leq 1$



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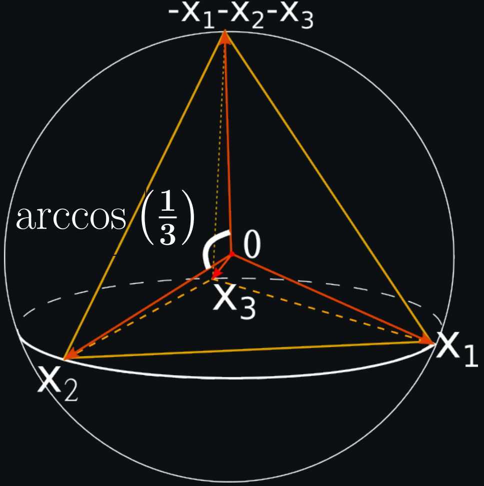
$$\|12 + 6 + 18\| \leq 1$$

Configuration of good triples



All good triples are concentrated in the shape of 3-simplex

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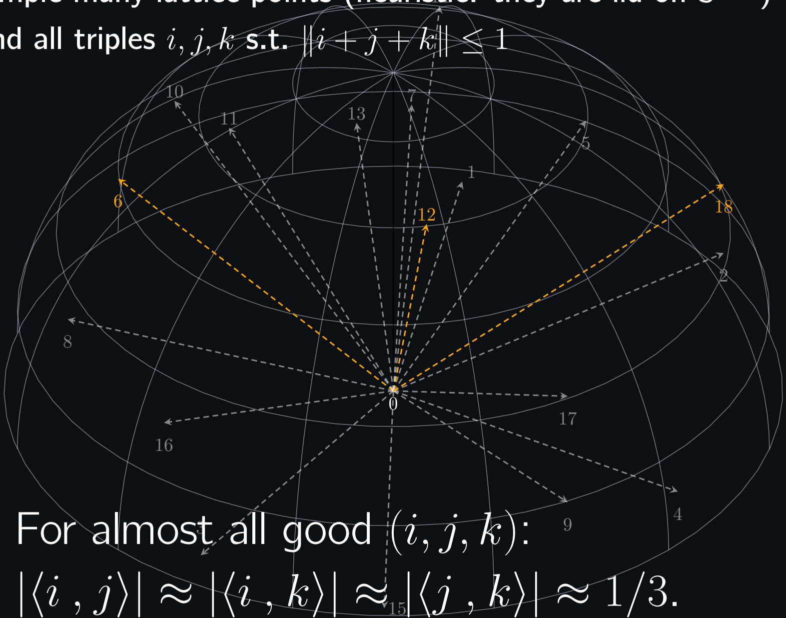

For almost all good (i, j, k):

$$|\langle i\ , j\rangle| \approx |\langle i\ , k\rangle| \approx |\langle j\ , k\rangle| \approx 1/3.$$

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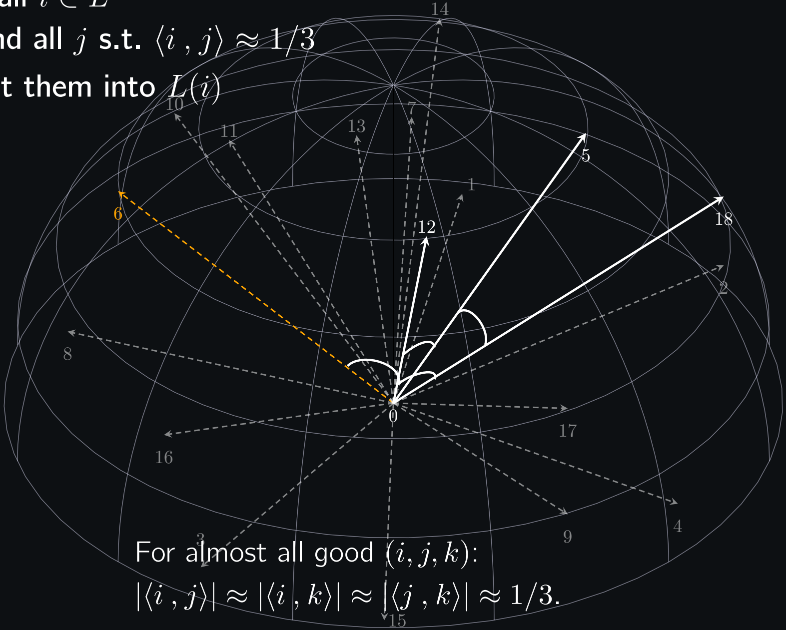
  Find all j s.t. $\langle i\,,\,j\rangle \approx 1/3$

  Put them into $L(i)$



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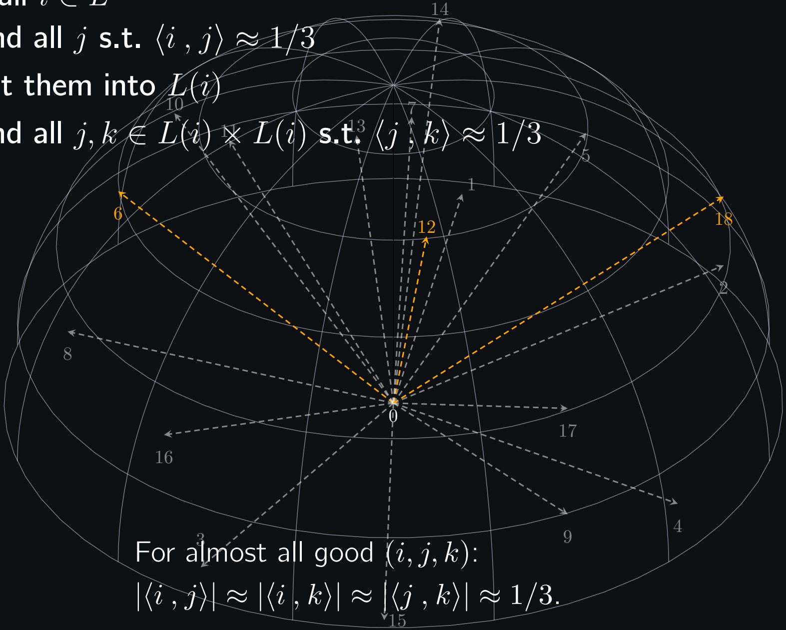

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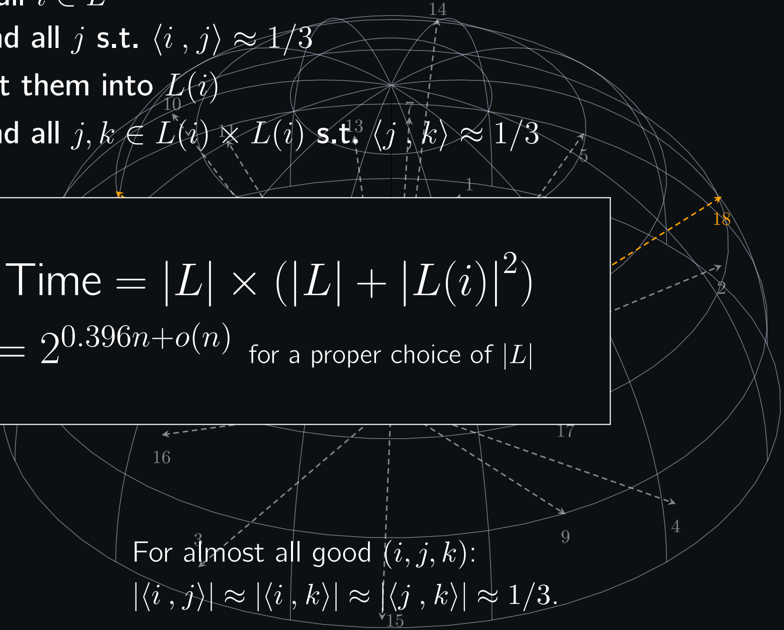

$$\text{Time} = |L| \times (|L| + |L(i)|^2)$$

$= 2^{0.396n+o(n)}$ for a proper choice of $|L|$

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Classical 3-Sieve

For all $i \in L$:

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$\quad$For all $j \in L(i)$

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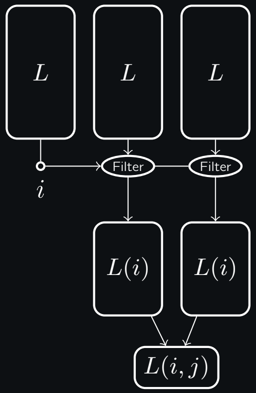

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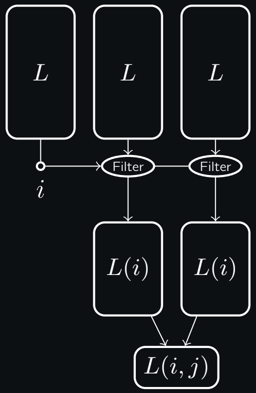

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Set $|L|$ s.t. we output $\approx |L|$ triples

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$$\text{TimeQ} = |L| \times \left(2\sqrt{\frac{|L|}{|L(i)|}} \cdot |L(i)| \right) = 2^{0.335n}$$

$$\text{TimeC} = |L| \times (|L| + |L(i)|^2) = 2^{0.396n}$$

Quantum k-Sieve

The algorithm generalises to larger $k = \Theta(1)$ and time-optimal inner product leading to

Time = $2^{0.299n+o(n)}$ Memory = $2^{0.139n+o(n)}$

Asymptotically time-optimal algorithm uses hashing techniques for $k = 2$ and achieves (T.Laarhoven)

Time = $2^{0.265n+o(n)}$ Memory = $2^{0.265n+o(n)}$

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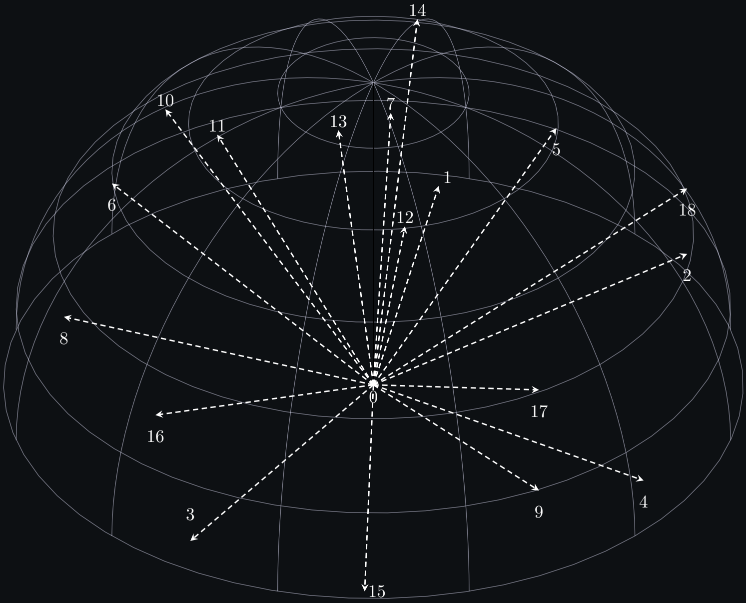
Connect two points $(i, j) \Leftrightarrow |\langle i\ , j \rangle| \approx 1/3$



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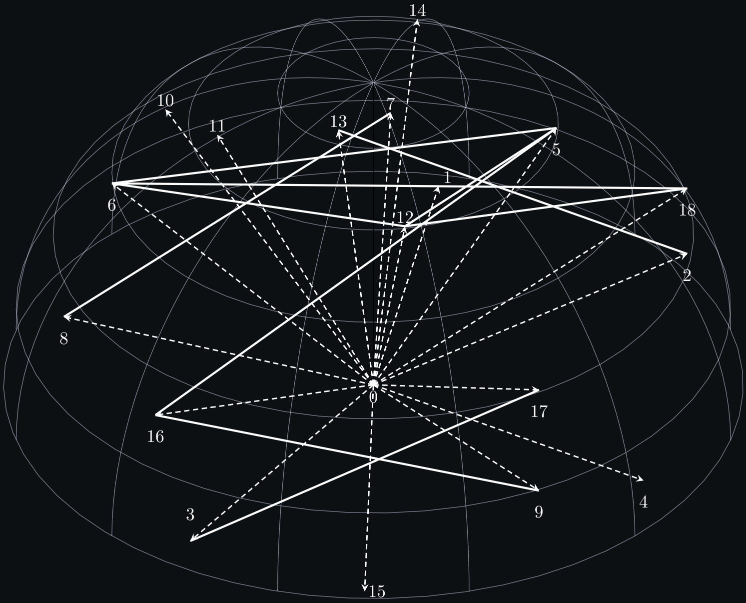

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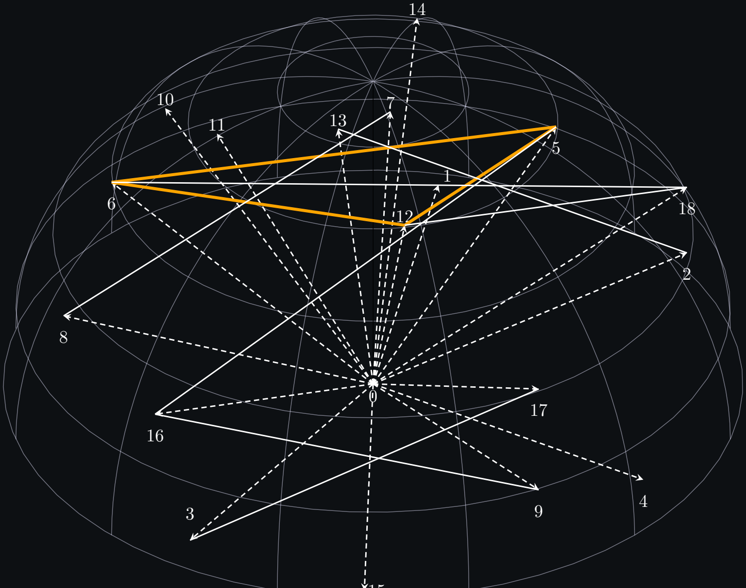
Good triples $(i, j, k) \Leftrightarrow$ triangles



Apply quantum triangle (k-clique) finding

$G = \{V, E\}$, V – lattice vectors, $e(v_i, v_j) \in E \Leftrightarrow |\langle v_i\,, v_j\rangle| \approx 1/3$

Run triangle listing on G (it's a sparse graph!)

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Adapt the triangle **finding** algorithm of [Buhrman–de Wolf-Dürr–Heiligman–Høyer–Magniez–Santha]:

$$\text{Time (find } \triangle) = \sqrt{|E|} \implies \text{Time (list all } \triangle's) = |V|\sqrt{|E|}$$

Gives the same runtime complexity as the previous algorithm for any $k = \Theta(1)$.

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Task: Given a large list L of vectors find all (i, j) s.t.

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[Beals–Brierley–Gray–Harrow–Kutin–Linden–Shepherd–Stather]:

For a list L and $|L|$ functions

$$f_i(j) = \begin{cases} 1, & |\langle L[i]\,, L[j] \rangle| \approx 1/2 \\ 0, & \text{else} \end{cases}$$

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There exists a quantum circuit that implements 2-Sieve of width $2^{0.2075n+o(n)}$ and depth $2^{0.1037n+o(n)}$.