Overview of Quantum Cryptanalysis of Lattice Systems

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based on joint works with G.Herold, T. Laarhoven

Quantum Cryptanalysis of Post-Quantum Cryptography
The Simons Institute for the Theory of Computing
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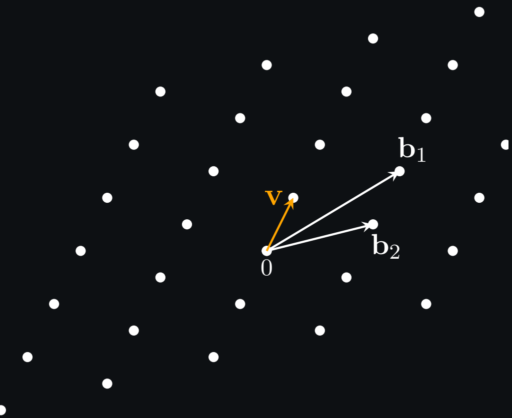
Outline

- The Shortest Vector Problem
- Classical & Quantum Sieve
- Other algorithms

SVP

Minimum

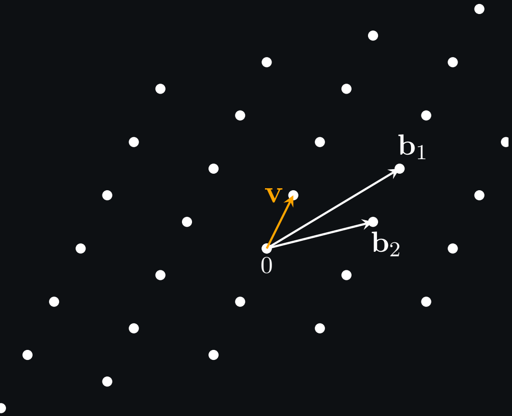
$$\lambda_1(\mathcal{L}) = \min_{\mathbf{v}\in\mathcal{L}\setminus\mathbf{0}} \|\mathbf{v}\|$$



The Shortest Vector Problem (SVP) asks to find $\mathbf{v}_{\text{shortest}} \in \mathcal{L}$:

$$\|\mathbf{v}_{\text{shortest}}\| = \lambda_1(\mathcal{L})$$

SVP



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$$\det(\mathcal{L}) = |\det(\mathbf{b}_i)_i|$$

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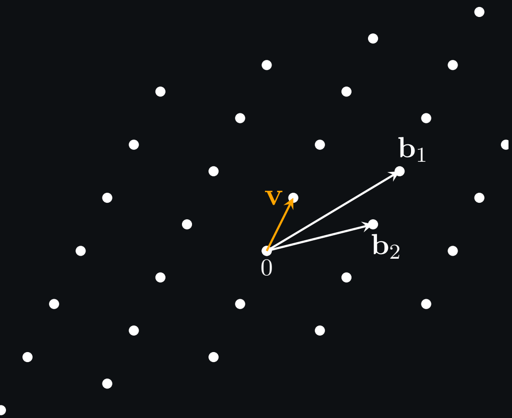

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Minkowski bound

$$\lambda_1(\mathcal{L}) \le \sqrt{n}(\det(\mathcal{L}))^{\frac{1}{n}}$$

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Asymptotics for SVP Algorithms

- Enumeration

- Sieving

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$$\text{Classical: Time} = 2^{((1/2e)+o(1))n\log n} \qquad \text{Mem.} = \mathrm{poly}(n)$$

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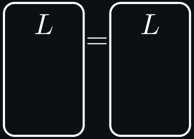
- Sieving (Heuristic)

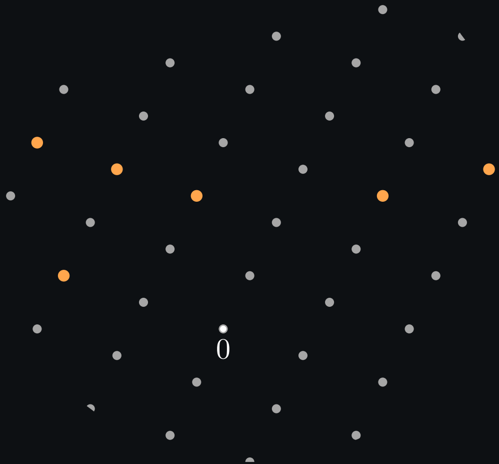
Classical: Time $= 2^{(0.292+o(1))n}$ Mem. $= 2^{(0.2075+o(1))n}$

Quantum: Time $= 2^{(0.265+o(1))n}$ Mem. $= 2^{(0.265+o(1))n}$

Basic 2-Sieve (Nguyen-Vidick sieve)

Main idea: Sample many Gaussian lattice vectors so that their sums give short(er) vectors





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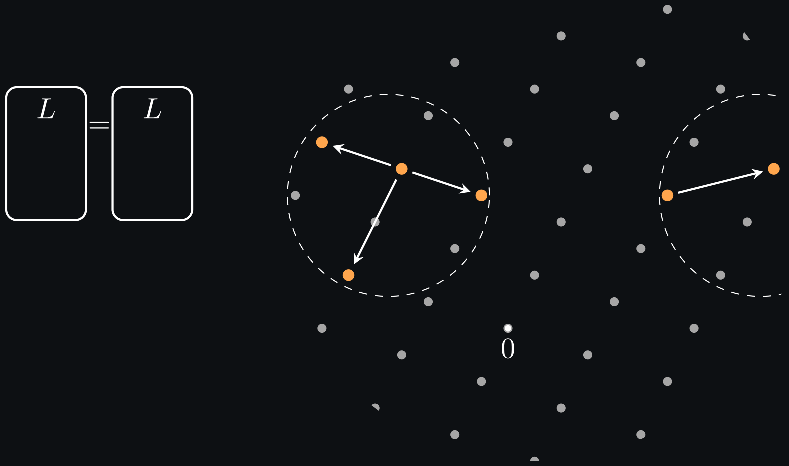

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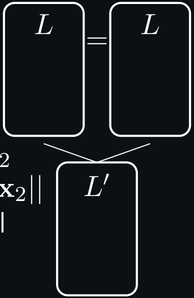

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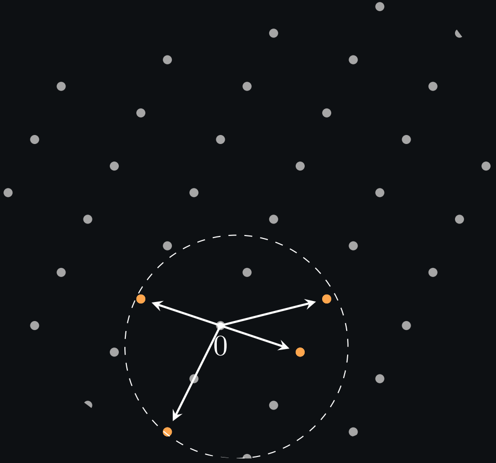
is small



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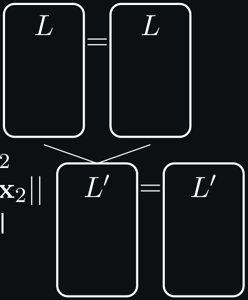

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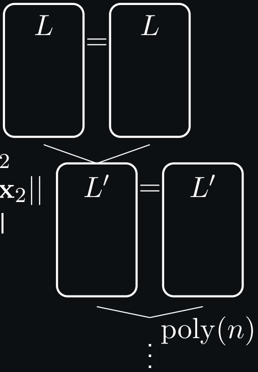
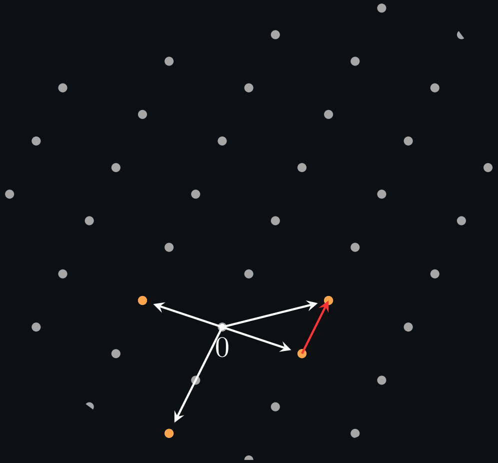
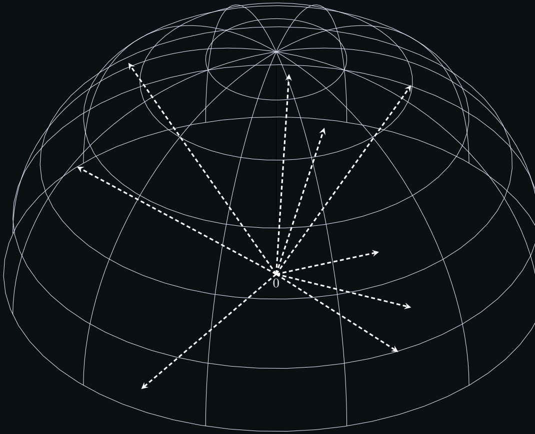




Main Routine in Sieving: 2-List problem on the unit sphere

Given 2-lists $L_1, L_2 \subset \mathcal{S}^{n-1}$ of iid. elements

$$\boxed{L_1} \quad \boxed{L_2} \quad \subset$$



find all $(x_1, x_2) \in L_1 \times L_2$:
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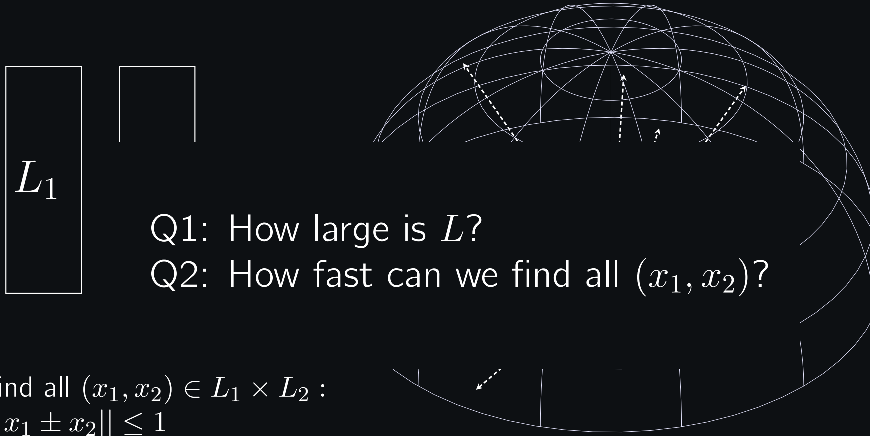

Q1: How large is L?

Q2: How fast can we find all (x_1, x_2)?

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Distribution of Gram matrices

Let $\mathcal{C} \in \mathbb{R}^{k \times k}$ be the Gram matrix of $x_1, \ldots, x_k \in \mathcal{S}^{n-1}$:

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- The Gram matrix $\mathcal{C}(x_1, \ldots, x_k)$ follows a distribution with density function

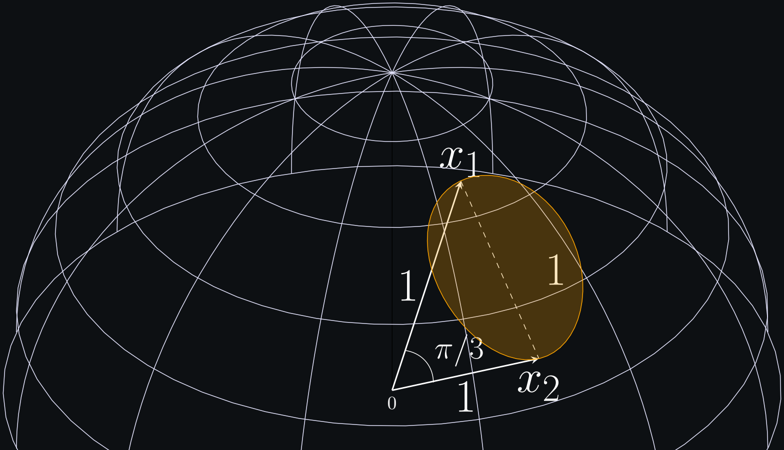
$$\mu_{\mathcal{C}} = \mathcal{O}(\det(\mathcal{C})^{\frac{1}{2}(n-k)}) \mathrm{d}C_{1,2} \ldots \mathrm{d}C_{k-1,k}$$

A proof is in [HK'17] and relies on the Wishart distribution

Q1: How large is L?

$$\mu_{\mathcal{C}} \approx \det(\mathcal{C})^{\frac{n}{2}} = \det \begin{pmatrix} 1 & \langle x_1 , x_2 \rangle \\ \langle x_1 , x_2 \rangle & 1 \end{pmatrix}^{\frac{n}{2}}$$

$$\det \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} = \frac{3}{4} \implies |L| = \left(\frac{4}{3}\right)^{n/2+o(n)} = 2^{(0.2075+o(1))n}$$



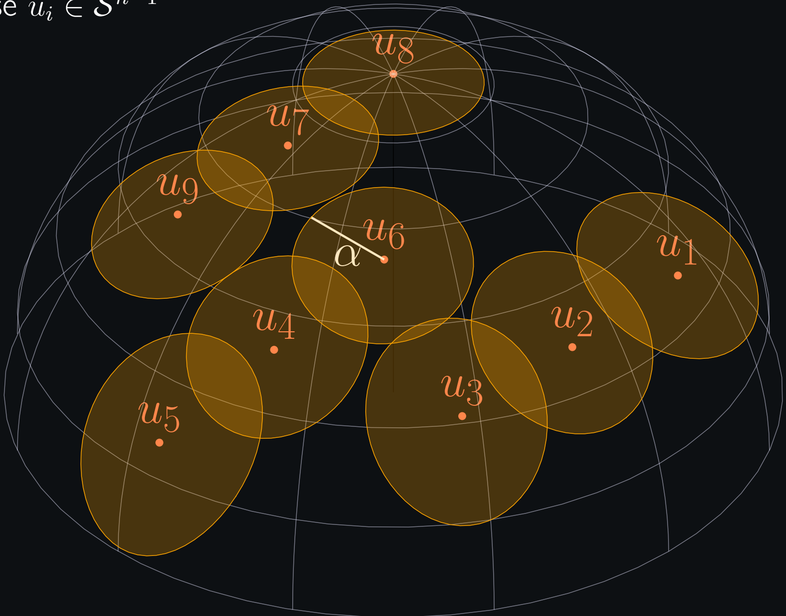
Q2: How fast can we find all (x_1, x_2)?

Brute force complexity: $|L|^2 = 2^{(0.415+o(1))n}$

To achieve $T = 2^{0.292+o(1)}$ use Near Neighbor search
(aka Locality-Sensitive techniques)

Locality-sensitive filtering [MO15, BGJ15, BDGL16]

choose $u_i \in \mathcal{S}^{n-1}$



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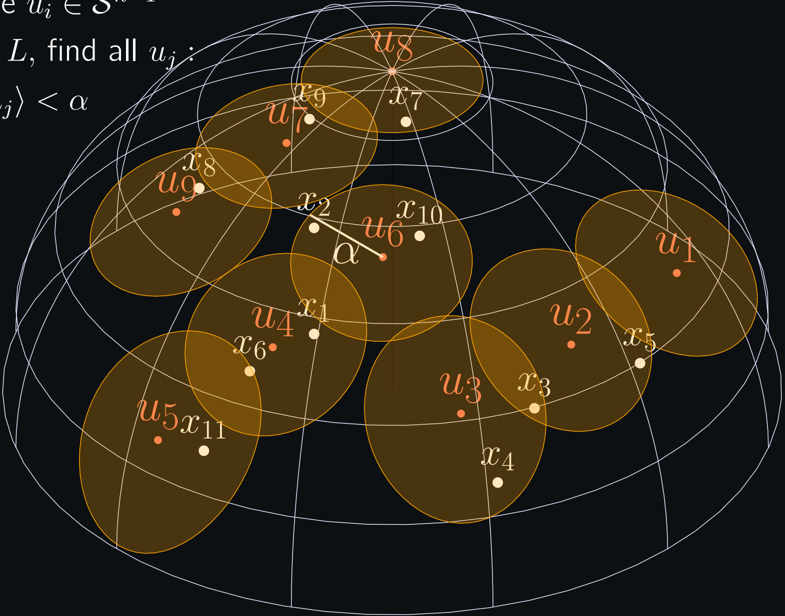
$\forall x_i \in L$, find all u_j :

$\langle x_i \,, u_j \rangle < \alpha$



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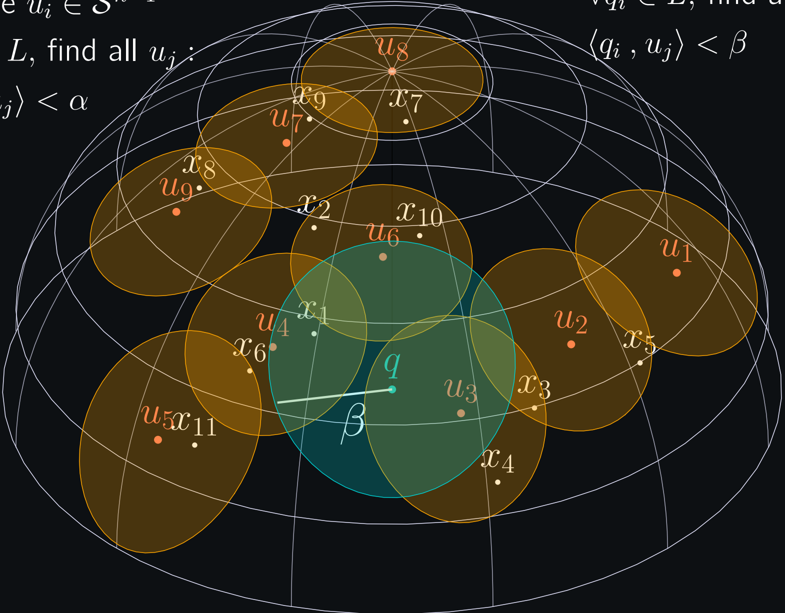
$\forall q_i \in L$, find all u_j :

$\langle q_i \, , u_j \rangle < \beta$



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   [MO15, BDGL16]: choose u from a product code.

   $$u \in C_1 \times C_2 \times \ldots \times C_t =: U,$$

   C_i - spherical codes of length $o(n)$.

   To obtain all close centers to $x = [x_1|x_2|\ldots|x_t]$, decode x_i wrt. C_i.

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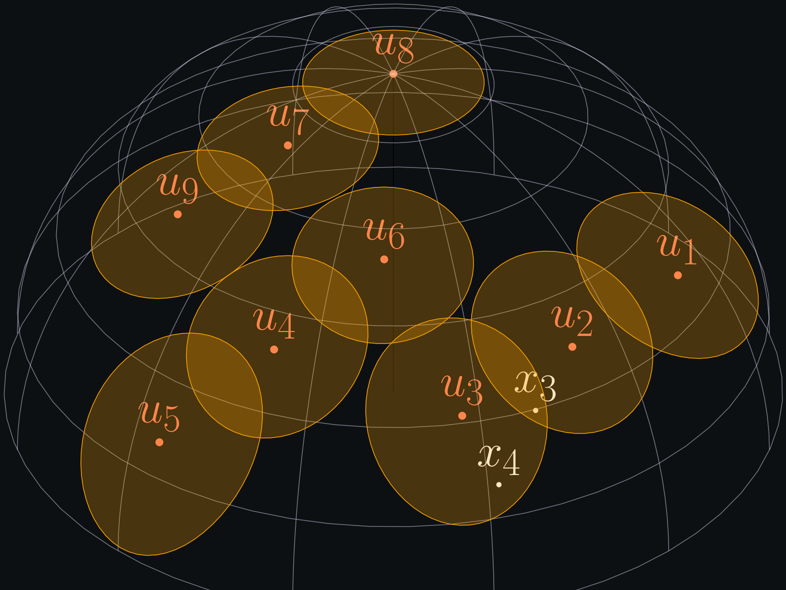
3. For each $x \in L$: query all relevant $u_i \in U$:
   $$T_{\text{Query}} = |U| \cdot \Pr_{u_i \in \mathcal{S}^{n-1}} [\langle u_i\, , x \rangle < \beta] = |U| \cdot (1-\beta^2)^{n/2}.$$

How large is U?



How large is U?

$|U|$ is determined by $P = 1/|U|$ that conditioned on $\langle x_3 , x_4 \rangle = 1/2$

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For $\alpha = \beta = c = 1/2,\ P^{-1} = \left(\frac{3}{2}\right)^{n/2} = 2^{0.292n}$

Quantum LSF [Laa'15]

Main idea: apply Grover search inside each bucket

Classical:

1. $T_{\text{Update}} = |U| \cdot (1 - \alpha^2)^{n/2}$
2. $T_{\text{Query}} = |U| \cdot (1 - \beta^2)^{n/2}$
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Quantum:

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$$T^{\text{Q}}_{\text{Query}} = |U| \cdot (1-\beta^2)^{n/2} + \sqrt{|\#\text{ relevant centers }| \cdot |\text{Bucket size}| \cdot |\#\text{ solutions}|} = |U| \cdot (1-\beta^2)^{n/2} + \sqrt{|U| \cdot (1-\beta^2)^{n/2} \cdot |L| \cdot (1-\alpha^2)^{n/2} \cdot |L| \cdot (1-c^2)^{n/2}}$$

$T^{\text{Q}}_{\text{Query}} = T^{\text{Q}}_{\text{Update}}$ for $\alpha = \frac{\sqrt{3}}{4}$ and $|U| = (\frac{13}{9})^{n/2} = 2^{0.265 \cdot n}$

Other Ways to Attack SVP/LWE/SIS?

1. Low-memory SVP: Enumeration

   Classical : depth-first traversal of a tree of size

   $$T_{\mathsf{Enum}} = 2^{((1/2e)+o(1))n\log n}$$

   Quantum: back-tracking technique [ANS18]

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Thank you!

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